Problem 1.

For a fixed odd natural number $n \geq 3$, find all functions $f_n : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f_n(x+y) = f_n^n(x) + \sqrt[n]{f_n(y)}$$

for all $x, y \in \mathbb{R}$.

Marking scheme:

Points for logical blocks are summed. A solution is considered complete (10 points) if all logical blocks are completed and the correct answer is obtained. Deductions for typical mistakes are applied to the total points after summation.

1. Stating the correct answer without proof or with an incorrect proof. **(1 point)**
2. **Logical blocks of the proof:**
   (a) Complete proof that $f_n(0) = 0$. **(3 points)**
   (b) Proof that $E(f_n) = \{-1; 0; 1\}$. **(3 points)**
   (c) Proof that $f(x_0) \neq \pm 1$ at any point (by contradiction). **(3 points)**
3. In alternative solutions not using the fact $f_n(0) = 0$: complete proof that $f_n(x) \equiv const$. **(6 points)**

Typical mistakes:

- Using even/odd properties of f: **-2 points**
- Missing -1 in $E(f_n)$ (squaring negative parts): **-2 points**
- Solving via the fact that $f_n(x) \equiv -1$; 1 and 0: **-3 points** (block 2 not counted)
- Using differentiability of the function: **-3 points**

Problem 2.

Let A be a real matrix of size $n \times n$. We define its rotation by 90° counterclockwise as A^r. An example of the operation is as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^r = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}.$$

Let $A^r + B^r = B^r A^T$ (where A^T is the transpose of A). Prove that $AB = BA$.

Marking scheme:

Points for items 1–5 are not added together. Inaccuracies in the proofs of items 6 and 7 deduct points from the final score proportionally to the number of inaccuracies. Errors in the proof (in particular, its incompleteness) of items 2 and 3 nullify the points for the corresponding items.

1. The property $(A + B)^r = A^r + B^r$ of the rotation operator is given. **(1 point)**
2. Commutativity of matrices is proven for the case $n = 2$. **(2 points)**
3. Commutativity of matrices is proven for the case $n = 3$. **(3 points)**
4. Representation $A^r = SA^T$ is given, where $S_{ij} = \delta_{n+1-i,j}$. **(5 points)**
5. The statement $(AB)^r = B^r A^T$ is given. **(6 points)**
6. Using statements from items 4 or 5, the equality $A + B = AB$ is established. **(+1 point)**
7. From the correctly proven property $A + B = AB$, commutativity of matrices is established. The problem is fully solved. **(+4 points)**

Problem 3.

Consider 4 spheres pairwise intersecting such that no two are tangent. Assume that no three spheres share a common circle of intersection. For each pair of spheres, consider the unique plane containing their circle of intersection (6 planes total). Prove that either all 6 planes are concurrent (intersect in a common point), or all 6 planes are parallel to a single fixed line.

Marking scheme:

1. Any example is given or it is proven that planes perpendicular to the edges of a trihedral angle intersect at one point. **(1 point)**
2. Only the case of four centers in one plane is considered, or the case of three centers is considered but intersection along one line is not proven, or the case of parallel planes for three spheres is missed. **(2 points)**
3. The statement from item 2 is proven. That is, all cases are considered and strictly justified. **(3 points)**
4. Correct approach, but 2-3 minor statements are without proof, or in an analytical solution the case $\det A = 0$ is incorrectly handled. **(7 points)**
5. Correct approach, but 1 minor statement is without proof, or the case of 4 centers in one plane is missed. **(8 points)**
6. Minor oversight in the proof. **(9 points)**
7. Completely correct solution. **(10 points)**

Problem 4.

Given a natural number n, what is the largest number of $n \times n$ integer matrices that can be chosen such that for any nonempty subset of these matrices, the determinant of their sum is odd?

Marking scheme:

Omission of key details in proofs or insufficiently rigorous justification deducts points from the final score proportionally to the number of inaccuracies.

1. The case $n = 1$ is correctly considered and an example is given, and an example with two matrices for $n = 2$ is given, but without proof that 3 matrices cannot be taken. **(1 point)**
2. The inequality "number of matrices" $\leq$ "matrix size" is proven. **(3 points)**
3. Completely correct solution. **(10 points)**

Problem 5.

Let $y(x)$ and $z(x)$ be solutions, defined in a left neighbourhood of zero and tending to $+\infty$ as $x \to -0$, of equations

$$y^{(20)}(x) = y^{25}(x) \quad \text{and} \quad z^{20}(x) = z^{(25)}(x), \quad \text{respectively.}$$

1. For some particular solutions, calculate the limit

$$\lim_{x \to -0} \frac{\ln y(x)}{\ln z(x)}.$$

2. Find all possible values of this limit.

Marking scheme:

Points for subitems (a) and (b) are summed. The total score for the problem is the sum of points for the first and second items.

1. **First item**:
   (a) Particular solutions are given without justification and the limit of the logarithm ratio is computed. **(+2 points)**
   (b) Particular solutions are justified. **(+1 point)**
2. **Second item**:
   (a) It is shown that all derivatives of the solutions tend to infinity as $x \to -0$. **(+3 points)**
   (b) The solution is carried through to the end. **(+4 points)**

Problem 6.

Let a continuous function $f : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ be given. For arbitrary $a, b > 0$, we determine the coordinates of the center of mass of a rectangle $(0, a) \times (0, b)$ with density $f(x, y)$ as follows:

$$g_x(a,b) = \frac{\int_0^a \int_0^b x f(x,y)\,dy\,dx}{\int_0^a \int_0^b f(x,y)\,dy\,dx}, \quad g_y(a,b) = \frac{\int_0^a \int_0^b y f(x,y)\,dy\,dx}{\int_0^a \int_0^b f(x,y)\,dy\,dx}.$$

Let $p > 0$, $q > 0$ be some fixed constants.

1. Find all functions $f(x, y)$ such that

$$g_x(a,b) = \frac{a}{p+1}, \quad g_y(a,b) = \frac{b}{q+1}.$$

2. For continuous positive functions $h(b)$ and $k(a)$, let the coordinates of the center of mass have the form

$$g_x(a,b) = \frac{a}{p+1} \cdot h(b), \quad g_y(a,b) = \frac{b}{q+1} \cdot k(a).$$

   Find necessary and sufficient conditions on $h(b)$ and $k(a)$ under which such a continuous positive function $f(x, y)$ exists. Explicitly specify all such functions $f(x, y)$.

Marking scheme:

1. Considering very special cases ($f(x, y) = 1$). **(1 point)**
2. Stated solution in the first part (evaluated based on neatness and justification). **(2–3 points)**
3. Searching for $f(x, y)$ in the form of a product of functions of x and y and neatly deriving solutions. **(3 points)**
4. Deriving a particular solution of the first part using differential equations. **(4 points)**
5. Solution of the second part not carried through to the end, evaluated based on progress. **(5–7 points)**
6. Complete solution. **(10 points)**