RUDN Math Olymp 2023

Problem 1.

Let $A, B \in M_{n\times n}(\mathbb{R})$ be symmetric martices such that

$$(AB + BA - A^2 - I_n)^2 = AB^2 - B^2A.$$

Find $rank(A^2 + B^2)$.

Problem 2.

Consider a hyperbola on a plane with no coordinate system given. Construct the center of symmetry of this hyperbola (a point O, having the following property: each point symmetric to a point of a figure with respect to O is a point of this figure itself) using only a straightedge (infinite length, has only one edge, and no markings on it) and a compass (noncollapsing compass with no maximum or minimum radius). Describe the construction algorithm and substantiate its validity.

Problem 3.

Compute the 2023^{rd} derivative of the arctangent function:

$$\frac{d^{2023}}{dx^{2023}} \arctan(x).$$

Problem 4.

Consider the equation

$$y' - y^2 + \frac{3my}{x} - \frac{2m^2}{x^2} = 0, \quad x \in [1, +\infty).$$

a) Find the general solution of this equation for $m = 1$.

b) Inspect if there exists a solution defined at all points of $[1, +\infty)$ for $m = 1$.

c) Find all m such that the equation has at least one solution defined at all points of $[1, +\infty)$.

Problem 5.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a triply continuously differentiable function such that the following conditions are satisfied:

$$f(x) < 0; \quad f'(x) > 0; \quad f''(x) < 0; \quad f'''(x) > 0; \quad \forall x \in \mathbb{R},$$

and, in addition:

$$27 \cdot f'''(x) + f(x) \leqslant 0.$$

a) Prove that the following inequality holds:

$$3 \cdot f'(x) + 2 \cdot f(x) < 0, \quad \forall x \in \mathbb{R}.$$

b) Give an example of a function that satisfies all of the mentioned conditions.